

Elegant Connections in Physics: Quantized Angular Momentum, ‘g-factors,’ Precessions, and Exchange Symmetry

— by Dwight E. Neuenschwander

In our study of the vector algebra used in general physics, we learned that the cross product of a vector with itself vanishes: $\mathbf{A} \times \mathbf{A} = 0$. However, quantum mechanics shows an exception to this “obvious” identity, in the peculiar case of angular momentum \mathbf{L} ,

$$\mathbf{L} \times \mathbf{L} = i\hbar \mathbf{L}, \quad (1)$$

which vanishes only in the classical limit $\hbar \rightarrow 0$. This weird property of *quantized* angular momentum is inherited from a peculiar interpretation of *linear* momentum, which itself emerges as an immediate consequence of the wave-particle duality postulates of deBroglie, Planck, and Einstein.

The bizarre cross product of Eq. (1) succinctly summarizes the commutation relations that exist among L_x , L_y , and L_z , in particular,

$$[L_\alpha, L_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} L_\gamma \quad (2)$$

(α , β , and γ stand for x , y , and z and $\epsilon_{\alpha\beta\gamma} = +1(-1)$ if $\alpha\beta\gamma$ is an even (odd) permutation of xyz and $\epsilon_{\alpha\beta\gamma} = 0$ if any two indices are equal). We also learned in our quantum courses that every component commutes with the square of the angular momentum vector itself:

$$[L^2, L_\alpha] = 0. \quad (3)$$

Where do these commutation relations come from? They come from merging the familiar definition of *angular* momentum with the quantum interpretation of *linear* momentum! First, we recall the familiar definition of \mathbf{L} : if a particle has linear momentum \mathbf{p} , and is located by the position vector \mathbf{r} relative to some point O , then that particle’s angular momentum about O is defined by

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}. \quad (4)$$

Meanwhile, according to quantum mechanics, whenever a particle’s momentum \mathbf{p} appears in a formula we can replace \mathbf{p} with $(\hbar/i)\nabla$. Where does *this* weird assertion come from? It goes back to the deBroglie and Planck-Einstein hypotheses, the basic postulates of quantum mechanics: Corresponding to a *free particle* of momentum \mathbf{p} and energy E , there exists a *harmonic wave* of wavenumber k and angular frequency ω , where, numerically, the particle properties are related to the wave properties by Planck’s constant \hbar :

$$E = \hbar\omega \quad (5)$$

and

$$\mathbf{p} = \hbar\mathbf{k}. \quad (6)$$

Any harmonic ψ moving along the x -axis always contains the factor

$$\psi \sim \exp[i(kx \pm \omega t)] \quad (7)$$

which, according to the deBroglie and Planck-Einstein hypotheses,



Wolfgang Pauli and Niels Bohr in 1954, demonstrating the ‘tippe top’ at the inauguration of the new Institute of Physics at Lund, Sweden.

Credit: Photograph by Erik Gustafson, courtesy AIP Emilio Segrè Visual Archives, Margrethe Bohr Collection

carries information about the particle’s dynamical variables according to

$$\psi \sim \exp[i(px \pm Et)/\hbar] \quad (8)$$

where i denotes $\sqrt{-1}$. So if we need to evaluate $p\psi$, we see this to be equivalent to evaluating the derivative $(\hbar/i) \partial\psi/\partial x$. Therefore, for $\mathbf{L}\psi$ we write in quantum language $\mathbf{L}\psi = (\hbar/i) \mathbf{r} \times \nabla\psi$. From this humble but strange beginning, the commutation relations of Eqs. (1-3) are deductive consequences. And from those commutation relations there follow all sorts of wonderful consequences about matter and radiation, from nuclei and atoms to semiconductors and proteins and neutron stars—our macroscopic world, flooded by light and built of matter that does *not* automatically collapse, owes much to the fact that the right-hand side of Eq. (1) is *not precisely* 0. To begin *that* story, we will see how the commutation relations show that angular momentum *must* come quantized in units that are integer multiples of $\frac{1}{2}\hbar$.

In working out some applications of quantized angular momentum, we encounter two pairs of topics that are distinct yet similar enough to be confusing if one only hears about them in passing. These are (1) two “g-factors—” the gyromagnetic ratio, typically denoted g ; and the Landé g -factor, another g ; and (2) Larmor precession with its “Larmor frequency,” and Thomas precession with its “Thomas frequency.” They enter atomic physics through the same problems, but for different reasons.

We shall pursue of these topics in this article, beginning with *why* angular momentum is quantized in units of $\frac{1}{2}\hbar$. We will conclude by recalling some deeper questions about *spin* angular momentum.

WHY ANGULAR MOMENTUM IS QUANTIZED IN UNITS OF $\frac{1}{2}\hbar$

Even though a vector in physical space is supposed to be described by *three* numbers, the commutation relations for \mathbf{L} mean that we can *simultaneously* measure for this vector, at most, only *two* numbers. They are its magnitude $|\mathbf{L}|$ (through L^2) and one (but *only* one) component, which for definiteness we will designate L_z . A system that carries quantized angular momentum will be described by some state vector $|\psi\rangle$ that is an eigenvector of both L^2 and L_z . Let their respective eigenvalues be λ and μ , so that

$$L^2 |\psi\rangle = \lambda |\psi\rangle \quad (9)$$

and

$$L_z |\psi\rangle = \mu |\psi\rangle. \quad (10)$$

For our first job we must find λ and μ . Because Planck's constant \hbar carries the dimensions of angular momentum, dimensional analysis requires that λ must be proportional to \hbar^2 and μ must be proportional to \hbar .

For the purpose of determining λ and μ (and we must admire the cleverness of whoever first came up with this) we define the quantities

$$L_{\pm} = L_x \pm iL_y. \quad (11)$$

It's the components of angular momentum that we *can't* observe that hold the key to finding out what we *can* observe! From the commutation relation of Eq. (3) we have at once

$$[L^2, L_{\pm}] = 0 \quad (12)$$

which means that the state

$$L_{\pm} |\psi\rangle \equiv |\psi_{\pm}\rangle \quad (13)$$

is also, like $|\psi\rangle$, an eigenstate of L^2 , but with some as-yet-unknown eigenvalue λ' ,

$$L^2 |\psi_{\pm}\rangle = \lambda' |\psi_{\pm}\rangle. \quad (14)$$

OK then, because L^2 and L_{\pm} commute, $\lambda = \lambda'$, as can be easily demonstrated:

$$\begin{aligned} L^2 |\psi_{\pm}\rangle &= L^2 (L_x \pm iL_y) |\psi\rangle \\ &= (L_x \pm iL_y) L^2 |\psi\rangle \\ &= (L_x \pm iL_y) \lambda |\psi\rangle \\ &= \lambda |\psi_{\pm}\rangle. \end{aligned} \quad (15)$$

And from the commutation relation of Eq. (2) it follows that

$$[L_z, L_{\pm}] = \pm \hbar L_{\pm}. \quad (16)$$

Consider then the effect of operating on $|\psi_{\pm}\rangle$ with L_z :

$$\begin{aligned} L_z |\psi_{\pm}\rangle &= L_z L_{\pm} |\psi\rangle \\ &= (L_{\pm} L_z \pm \hbar L_{\pm}) |\psi\rangle \\ &= (\mu \pm \hbar) |\psi_{\pm}\rangle. \end{aligned} \quad (17)$$

This means that the operators L_+ and L_- create a set of “rungs” in a “ladder” of angular momentum states, where each differs from its neighbor by one quantum \hbar . Evidently we can write the eigenvalue equation

$$L_z |\psi_{\pm}\rangle = \mu' |\psi_{\pm}\rangle \quad (18)$$

where $\mu' = \mu \pm \hbar$. What a beautiful result!— L_+ raises and L_- lowers the L_z eigenvalue of $|\psi\rangle$ by one quantum, \hbar .

Because the component of a vector cannot exceed in magnitude the vector itself, so that in our case $|L_z| \leq |\mathbf{L}|$, there must exist a state with a maximum L_z , call it $|\Psi_{\text{top}}\rangle$, such that

$$L_+ |\Psi_{\text{top}}\rangle = 0. \quad (19)$$

Let $|\Psi_{\text{top}}\rangle$ have L_z eigenvalue μ_{top} , which says

$$L_z |\Psi_{\text{top}}\rangle = \mu_{\text{top}} |\Psi_{\text{top}}\rangle. \quad (20)$$

From dimensional analysis, it must happen that

$$\mu_{\text{top}} = \ell \hbar \quad (21)$$

for some dimensionless number ℓ . So now we must find ℓ . Because λ depends only on L^2 and not upon L_z , we may write for $|\Psi_{\text{top}}\rangle$

$$L^2 |\Psi_{\text{top}}\rangle = \lambda |\Psi_{\text{top}}\rangle. \quad (22)$$

Let's write the left-hand side of this as

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 \\ &= L_{\pm} L_{\mp} \mp \hbar L_z + L_z^2. \end{aligned} \quad (23)$$

Using the lower sign from Eq. (23), Eq. (22) becomes

$$(L_- L_+ + \hbar L_z + L_z^2) |\Psi_{\text{top}}\rangle = \lambda |\Psi_{\text{top}}\rangle \quad (24)$$

Each term on the left side we know how to evaluate, so that

$$(0 + \hbar^2 \ell + \hbar^2 \ell^2) |\Psi_{\text{top}}\rangle = \lambda |\Psi_{\text{top}}\rangle \quad (25)$$

and thus

$$\lambda = \hbar^2 \ell(\ell + 1). \quad (26)$$

Likewise, there must exist a bottom “rung” $|\Psi_{\text{bottom}}\rangle$ with L_z eigenvalue $\hbar \ell'$ such that

$$L_- |\Psi_{\text{bottom}}\rangle = 0 \quad (27)$$

for some dimensionless number ℓ' . Since $|\Psi_{\text{bottom}}\rangle$ will have the same eigenvalue of L^2 as all the other angular momentum states on the “ladder,” we have

$$L^2 |\Psi_{\text{bottom}}\rangle = \lambda |\Psi_{\text{bottom}}\rangle. \quad (28)$$

So that we can make use of Eq. (27), employ the upper sign from Eq. (23) in Eq. (28),

$$(L_+ L_- - \hbar L_z + L_z^2) |\Psi_{\text{bottom}}\rangle = \lambda |\Psi_{\text{bottom}}\rangle \quad (29)$$

which gives on the left-hand side

$$\hbar^2 \ell' (\ell' - 1) |\Psi_{\text{bottom}}\rangle = \lambda |\Psi_{\text{bottom}}\rangle. \quad (30)$$

Because it's the same λ for $|\Psi_{\text{top}}\rangle$ and $|\Psi_{\text{bottom}}\rangle$, it follows that

$$\hbar^2 \ell' (\ell' - 1) = \hbar^2 \ell (\ell + 1) \quad (31)$$

which requires either

$$\ell' = \ell + 1 \quad (32)$$

or

$$\ell' = -\ell. \quad (33)$$

The first alternative must be eliminated physically because the lowest rung of the angular momentum state ladder cannot have a larger value of L_z than the highest one! This leaves $\ell' = -\ell$: the bottom rung of the ladder is symmetric with the top rung.

Let's sum up what we have so far:

$$L^2 |\Psi\rangle = \hbar^2 \ell(\ell + 1) |\Psi\rangle, \quad (34)$$

$$L_z |\Psi_{\text{top}}\rangle = \hbar \ell |\Psi_{\text{top}}\rangle, \quad (35)$$

$$L_z |\Psi_{\text{bottom}}\rangle = -\hbar \ell |\Psi_{\text{bottom}}\rangle, \quad (36)$$

where

$$L_z |\Psi_{\pm}\rangle = (\mu \pm \hbar) |\Psi_{\pm}\rangle, \quad (37)$$

So it seems that a particle that has angular momentum \mathbf{L} will have quantized z -components with labels from $+\ell$ to $-\ell$, separated by integer steps. Define

$$m_{\ell} = \ell, \ell - 1, \dots, -\ell. \quad (38)$$

This means that ℓ must, itself, be either an integer or a half-integer. Were it not, then we could not have $\ell' = -\ell$. For instance, one might imagine that, say, $\ell = 3.1$, and count backwards by integer steps, then we would encounter the sequence

3.1
2.1
1.1
0.1
-0.9
-1.9,
-2.9
-3.9...

which skips over -3.1 . If ℓ is an integer or half-integer then there are $2\ell+1$ "rungs" in the "ladder" of z -component states. Let N denote the number of steps that will take us from the $-\ell$ state to the $+\ell$ state. Since the number of *steps* is one less than the number of *rungs*, we can say that

$$N = 2\ell \quad \text{where } N = 0, 1, 2, 3, \dots \quad (39)$$

Therefore, ℓ may assume these possible values:

$$\ell = \frac{1}{2}N = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots \quad (40)$$

We have demonstrated that when a system has angular momentum \mathbf{L} , and that system is subject to the rules of quantum mechanics, its

angular momentum must be described by a quantum state with eigenvalues ℓ and m_{ℓ} , so we denote the state as

$$|\Psi\rangle = |\ell, m_{\ell}\rangle \quad (41)$$

The angular momentum has magnitude $|\mathbf{L}|$ where

$$L^2 |\ell, m_{\ell}\rangle = \hbar^2 \ell(\ell+1) |\ell, m_{\ell}\rangle \quad (42)$$

with $\ell = 0, 1, 2, \dots$. The orientation of the \mathbf{L} vector relative to the z -axis can be in one of the directions given by

$$L_z |\ell, m_{\ell}\rangle = \hbar m_{\ell} |\ell, m_{\ell}\rangle \quad (43)$$

where $m_{\ell} = \ell, \ell-1, \dots, -\ell$. It will be noticed that the number of states of different m_{ℓ} , for each ℓ , is $2\ell+1$, the "multiplicity" of states having the same ℓ .

Our knowledge of the *direction* of \mathbf{L} is quite limited. According to classical mechanics, we can know the magnitude and all three components of \mathbf{L} simultaneously, to as many decimal places (in principle) as we wish. But for *quantized* angular momentum, we can know, in principle, the magnitude of \mathbf{L} but only one component of it (which we call the z -component). We have no clue about the x and y components of \mathbf{L} , so all we can say about the angular momentum vector is that it lies somewhere on the cone whose axis is the z -axis and the cone's surface makes the angle from the z -axis given by θ , which has a well-defined value when the particle is in a pure state $|\ell, m_{\ell}\rangle$:

$$\begin{aligned} \cos\theta &= \langle L_z \rangle / \sqrt{\langle L^2 \rangle} \\ &= m_{\ell} / \sqrt{\ell(\ell+1)} \end{aligned} \quad (44)$$

Because for a given value of ℓ the azimuthal quantum number m_{ℓ} takes on only $2\ell+1$ values, $\cos\theta$ has only $2\ell+1$ possible values. This is sometimes misleadingly called "space quantization;" I say "misleading" because its angular momentum, not space, that gets quantized. (Space *might* be quantized at the Planck scale 10^{-33} cm, but that's another story entirely!)

It's interesting to compare the circumstances of the quantization of angular momentum to those necessary for the quantization of energy. In the quantum world, a particle's energies are quantized only if the particle moves in some restricted region of space, so the deBroglie wave functions become standing waves of discrete frequencies. But when the particle is free, then its energy will be a continuous variable; the deBroglie waves are confined within no boundaries and may have any frequency whatsoever. But whether the particle is free or confined makes no difference to angular momentum: It's values of L^2 and L_z are necessarily quantized, and its values of L_x and L_y are not measurable. This occurs because the deeper relation $\mathbf{p} = (\hbar/i)\nabla$ does not depend on boundary conditions.

REPRESENTATIONS OF ANGULAR MOMENTUM

We recall that the angular momentum of a *system* of particles (in both the Newtonian or quantum paradigms) can be partitioned into the "orbital angular momentum" *of* the center of mass relative to some origin, plus the "spin angular momentum" *about* the center of mass.[1] Henceforth in this article, I will let \mathbf{L} denote the orbital angular momentum, \mathbf{S} the spin angular momentum, and \mathbf{J} their sum:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}. \quad (45)$$

For a Newtonian example, the Earth forms an extended body, orbiting the Sun while spinning on its axis. The Earth's total angular momentum \mathbf{J} can be partitioned into the angular momentum \mathbf{L} of the Earth's center of mass about the Sun, plus its spin angular momentum *about* the center of mass. \mathbf{L} is the Earth's angular momentum *as if* it were a single particle with no size or shape—a point mass. The details of its size and shape are carried by the spin angular momentum \mathbf{S} as the Earth spins on its axis.

Such a partition—at least a result analogous to it—also occurs for the electron in an atom. The orbital angular momentum states that go with \mathbf{L} will be denoted $|\ell, m_\ell\rangle$, and the spin angular momentum states that go with \mathbf{S} will be denoted $|s, m_s\rangle$. *Whatever* the values of ℓ (integer or half-odd integer), we have seen how the values of m_ℓ differ by integer steps, ranging from $-\ell$ to $+\ell$; and likewise, *whatever* the values of s (integer or half-odd integer), the values of m_s also differ by integer steps and range from $-s$ to $+s$. What *are* these values, and what do the states $|\ell, m_\ell\rangle$ and $|s, m_s\rangle$ look like explicitly? By that question I mean, what kinds of information do they contain, and how do we encode that information? (We will worry about the \mathbf{J} states later.)

When we neglect any details that may exist about the electron's size and shape (if it *has* a size and shape) and ignore its spin for the moment (as one does when deriving Kepler's laws for point-mass planets orbiting the Sun), we are then considering orbital angular momentum only. Consider the situation where the electron moves in the potential of a *central* force, that acts only in the radial dimension, and thus exhibits spherical symmetry about the origin. Central forces exert no torque and thus conserve orbital angular momentum. The particle's kinetic plus potential energy sums to the total,

$$\mathbf{p}^2/2m + U = E \quad (46)$$

which gets echoed in quantum mechanics in the time-independent Schrödinger equation for the energy eigenvalues E ,

$$-(\hbar^2/2m)\nabla^2\psi + U(r)\psi = E\psi. \quad (47)$$

Keep in mind that the orbital angular momentum \mathbf{L} is already built into Eqs. (46) and (47). How does *linear* momentum \mathbf{p} get turned into *angular* momentum \mathbf{L} ? It's through the kinetic energy, because both \mathbf{p} and \mathbf{L} depend on the orbiting particle having a velocity. First, note that because a central force exerts zero torque on the orbiter, $\mathbf{L} = \text{constant}$ in both magnitude and direction. So we can take the direction of \mathbf{L} to point along the z -axis, which means the particle orbits in the equatorial plane. Second, the linear momentum \mathbf{p} has radial component $m\mathbf{r}/dt$ and angular component $m\mathbf{r}d\phi/dt$. This means that $|\mathbf{L}| = |\mathbf{r} \times \mathbf{p}| = m r^2 d\phi/dt$. Finally, the kinetic energy $\mathbf{p}^2/2m$ or $\frac{1}{2}m\mathbf{v} \cdot \mathbf{v}$ may be written in terms of angular momentum:

$$\begin{aligned} \frac{1}{2}m \mathbf{v} \cdot \mathbf{v} &= \frac{1}{2}m(dr/dt)^2 + \frac{1}{2}mr^2 (d\phi/dt)^2 \\ &= \frac{1}{2}m(dr/dt)^2 + \frac{1}{2}L^2/mr^2. \end{aligned} \quad (48)$$

With a central potential energy (that does not depend on latitude, longitude, or time), in the spherical coordinates (r, θ, ϕ) of Fig. 1, the Schrödinger equation separates.

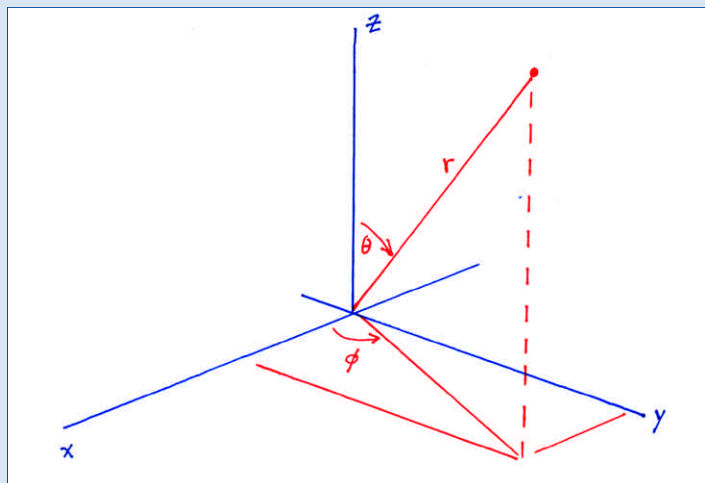


Fig. 1. Spherical coordinates.

To carry this out, one factors the wave function into radial and angular parts: $\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$. The Schrödinger equation neatly separates into one equation that has only r -dependence, and another equation having only the θ and ϕ dependence of latitude and longitude. The separation constants that enable the angular and radial parts to uncouple turn out to be none other than our now-familiar friend (but here obtained from another approach!):

$$\ell(\ell + 1) \text{ for } \ell = 0, 1, 2, 3, \dots$$

The angular equation separates a second time into an equation for θ and another equation for ϕ . The separation constant that enables the θ dependence to separate from ϕ is another of our friends,

$$m_\ell = \ell, \ell - 1, \dots, -\ell.$$

So in the case of a particle moving in *any* central potential, the orbital angular momentum states carry *integral* quantum numbers ℓ and their corresponding integers m_ℓ .

What do the corresponding orbital angular momentum *states* $|\ell, m_\ell\rangle$ look like *explicitly*? What information do they contain, besides the numerical values of ℓ and m_ℓ ? When you solve the partial differential equations for the latitude and longitude variables in the central potential problem (where the angular equation is *independent* of the potential), the solutions are the celebrated spherical harmonics, denoted $Y_\ell^{m_\ell}(\theta, \phi)$. They are combinations of sines and cosines of θ and ϕ . The spherical harmonics, as their name suggests, form an orthonormal basis for any function of latitude and longitude on a spherical surface. The spherical harmonics arise not only in quantum central force problems, but they appear ubiquitously in otherwise unrelated topics too, that take for their subject matter functions defined on spherical surfaces. Applications range from electrostatics to heliosismology. The spherical harmonics describe “standing waves” that may exist on a spherical surface, in terms of which any function of latitude and longitude can be expressed by their superposition.

Students of quantum theory (and general chemistry, which is *applied physics*) will recognize in the shapes of the hydrogen atom orbitals the first few spherical harmonics: on a sphere of any specific radius centered on the nucleus, the spherical harmonics for a given ℓ and m_ℓ describes how the probability density varies with latitude and

longitude when the system exists in that pure “standing wave” mode. Of course, at any moment the *actual* wave function for the electron in the hydrogen atom will be a superposition of the solutions we have just described, just as the actual wave on the cello string is some linear combination of its harmonic series of possible standing waves.

When the central potential in the Schrödinger equation happens to be the Coulomb interaction between a point nucleus and a single point electron, the radial equation can be solved *exactly* giving a set of functions $R(r)$ that depends on ℓ and another quantum number n introduced below. Each such function consists of a Laguerre polynomial, multiplied by an exponentially damped radial factor. One derives here a third quantum number, denoted n , that may take on the values $n = 1, 2, 3, \dots$. The electron’s energy levels in the Coulomb-only-interaction of a single electron in the hydrogen atoms are given by

$$\begin{aligned} E_n &= -\alpha^2 mc^2 (1/n^2) \\ &= -13.6\text{eV}/n^2, \end{aligned} \quad (49)$$

where

$$\alpha = ke^2/\hbar c \approx 1/137 \quad (50)$$

denotes the “fine structure constant,” a dimensionless combination of the fundamental constants from electrodynamics (the Coulomb constant k and the fundamental charge e), quantum theory (Planck’s constant), and special relativity (the speed of light). The fine structure constant sets the scale of the photon-charge interaction in quantum electrodynamics.

Back to the hydrogen atom: when working out the solution of this problem, it’s useful to organize the states by starting with n , which takes on the possible values $1, 2, 3, \dots$. If you know n , then the values of ℓ will be $0, 1, 2, \dots$ up to $n-1$, and when you know ℓ then the values of m_ℓ range in integer steps from $+\ell$ to $-\ell$. A moment’s reflection shows that for each n there are n^2 states all with the same energy in the Coulomb-only interaction model of the hydrogen atom.

But of course, there are more interactions between the orbiting electron and the proton than are captured by the electrostatic Coulomb interaction! The relative motion between the electron and proton means generates an internal magnetic field within the atom. This interaction reveals itself through the electron’s *spin* angular momentum.

The electron energy levels show up through their *differences*, in the energy spectrum of the photons that make transitions between those levels. However, there’s a subtle fine structure: each (n, ℓ, m) energy state seems to be *doubled* into a *pair* of states! We have not yet included the possibility of spin angular momentum for the electron, so if we now assume this doubling to be somehow involved with the electron’s spin, it appears that the electron’s spin angular momentum has a multiplicity of 2, that is,

$$2s + 1 = 2$$

so that

$$s = 1/2$$

and therefore

$$m_s = \pm 1/2.$$

On such considerations, spin- $1/2$ for the electron was proposed by Uhlenbeck and Goudsmit in 1926. Their hypothesis means that S_z has only two possible orientations, which we refer to casually as “spin up” and “spin down,” a spin polarization.

The magnetic field internal to the atom only *reveals* the spin; it does not *cause* spin. The spin of the electron is a property of the electron itself, not merely of its habitat. This can be shown by taking a

beam of *free* electrons and passing them through a slit followed by a magnetic field: the beam of electrons splits into two beams. When one tries to further split the separated beams again with magnets oriented the same way as the first magnet, no further splitting of the beams occur. The first magnet separates the electrons according to their two possible spin polarization states.

These two states, spin up and spin down, can serve as the basis states for a two-state system. Spin up and spin down “basis vectors” can be represented by a pair of two-component matrices:

$$|s, m_s\rangle = |1/2, +1/2\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \chi_+ \quad (51a)$$

and

$$|s, m_s\rangle = |1/2, -1/2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \chi_- \quad (51b)$$

Any possible spin state may be written as a superposition of these basis states:

$$\chi \equiv \begin{bmatrix} u \\ v \end{bmatrix} = u \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u \chi_+ + v \chi_- \quad (52)$$

where $|u|^2$ = probability that the electron will be found to have spin up, and $|v|^2$ = probability that it will be found to have spin down, when the spin is measured. In the case of the electron,

$$S^2 \chi_\pm = \hbar^2 s(s+1) \chi_\pm = 3/4 \hbar^2 \chi \quad (53)$$

and

$$S_z \chi_\pm = \pm 1/2 \hbar \chi_\pm \quad (54)$$

Once we write the spin state as a two-entry column matrix, then any manipulation of the spin, any way to query or modify the information contained in the spin matrices, must be represented with 2×2 matrices. Let’s construct a set of such matrices in terms of which the spin information can be manipulated.

From Eq. (53) we can see that a matrix representation for S^2 is

$$S^2 = 3/4 \hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (55)$$

Eq. (54) suggests the matrix representation for S_z as

$$S_z = 1/2 \hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (56)$$

We also recall the rungs of our “ladder” of states, where for the top and bottom rungs (the spin- $1/2$ ladder has only two rungs!),

$$S_+ \chi_+ = 0 \quad (57)$$

and

$$S_- \chi_- = 0. \quad (58)$$

In terms of matrices these relations will be satisfied by

$$S_+ = \hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (59)$$

and

$$S_- = \hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad (60)$$

But from the definition of $S_{\pm} = S_x \pm iS_y$ it follows that

$$S_x = \frac{1}{2}(S_+ + S_-) \quad (61)$$

and

$$S_y = -\frac{1}{2}i(S_+ - S_-) \quad (62)$$

Therefore

$$S_x = \frac{1}{2}\hbar \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (63)$$

and

$$S_y = \frac{1}{2}\hbar \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (64)$$

The spin vector \mathbf{S} for a spin- $\frac{1}{2}$ can now be represented as a “vector of matrices,”

$$\mathbf{S} = S_x \mathbf{i} + S_y \mathbf{j} + S_z \mathbf{k} = \frac{1}{2}\hbar \boldsymbol{\sigma} \quad (65)$$

where $\boldsymbol{\sigma}$ denotes the celebrated “Pauli spin matrices,”

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (66)$$

The mental picture of the electron as a wee little top is quite suggestive, of course; but it’s misleading if taken literally. Although the electron’s spin behaves algebraically like any other quantized angular momentum, and by that standard *is* definitely an angular momentum, it can’t be as simple as the familiar *mechanical motion* of a tiny spinning top. Here’s why: it’s a useful exercise to calculate the speed of a point on a spherical electron’s equator, if its radius is determined by setting its electrostatic energy equal to mc^2 , and setting the product of its moment of inertia and angular velocity equal to $\frac{1}{2}\hbar$. One finds that a point on the equator moves at something like 300 times the speed of light![2] Clearly a mechanical model cannot be taken literally (at least, not if the mass is associated with the electric energy density). *Whatever* the mechanism that *causes* spin, it seems to be an *intrinsic* property of each species of elementary particle. Indeed, one *definition* of an elementary particle has been offered as “a state of definite mass and spin.”[3] Electrons, quarks, and neutrinos are elementary particles that have $s = \frac{1}{2}$; photons have $s = 1$, gravitons (when they are detected) will have $s = 2$; the Higgs particles (or whatever in Nature plays the role of the conjectured Higgs particles) have $s = 0$. Composite objects such as protons and neutrons (bound states of quarks) or the helium-3 nucleus have odd-half-integer \hbar ; composites such as pions or the helium-4 nucleus have integer s . Particles with integer s are called “bosons,” and particles with odd-half-integer s are called “fermions.” When one encounters a system of two or more identical particles, it matters mightily whether they are bosons or fermions. More about that later.

If the elementary particle (or composite) carries some distribution of electric charge (even if it’s electrically neutral overall, like the neutron), then from the definition of the angular momentum of an extended body, and from the definition of a current distribution’s magnetic dipole moment, it follows that there should be, in general, a connection between spin and magnetic dipole moment. To that topic we turn next.

THE GYROMAGNETIC RATIO

Deep inelastic collisions show that the size (if such exists) of an

electron must be *smaller* than a scale on the order of 10^{-18}cm , about a hundred thousand times smaller than the proton. On the other hand, if the electron *really* is a genuine *particle*—a point mass—how can a *point* be endowed with a *dipole* moment (magnetic or otherwise)? Leaving those fascinating questions aside, as a *mechanical concept* the spin angular momentum would be a superposition of the sum of all the angular momenta of its parts distributed about the center of mass:

$$\begin{aligned} \mathbf{S} &= \int \mathbf{r} \times \mathbf{v} \, dm \\ &= \int \mathbf{r} \times \mathbf{v} \, \rho_{\text{mass}} \, dV \end{aligned} \quad (67)$$

where ρ_{mass} denotes the electron’s mass density and dV a volume element (if the electron were a genuine point mass, then the density would be the mass times the Dirac delta function). Similarly, because this spinning mass carries electric charge, its various parts make little current loops which generates a magnetic dipole.[4] When one does the multipole expansion for the magnetic field produced by a generic electric current distribution, the dipole field dominates. The magnetic dipole moment $\boldsymbol{\mu}$ emerges as a coefficient in the expansion:

$$\begin{aligned} \mathbf{m} &= \frac{1}{2} \int \mathbf{r} \times \mathbf{j} \, dV \\ &= \frac{1}{2} \int \mathbf{r} \times \mathbf{v} \, \rho_{\text{charge}} \, dV \end{aligned} \quad (68)$$

where \mathbf{j} denotes the current density and ρ_{charge} the charge density. It will be noticed that, aside from the $\frac{1}{2}$ (that comes from the Taylor series in the multipole expansion), the definitions of \mathbf{S} and $\boldsymbol{\mu}$ are almost identical, the only difference being that one has *mass* density and the other has *charge* density. If we divide the respective densities by the total charge and total mass, then take the ratio of these specific densities, we have a dimensionless number called the “gyromagnetic g -factor,” here denoted g_o :

$$g_o \equiv \frac{(\rho_{\text{charge}}/q)}{(\rho_{\text{mass}}/m)}. \quad (70)$$

In general, both densities may be a function of location; but if g_o does not vary throughout the electron’s volume, or if we are content with the ratio of average densities, then we see that

$$\mathbf{m} = g_o (q/2m)\mathbf{S}. \quad (71)$$

$g_o (q/2m)$ is called the “gyromagnetic ratio,” (denoted γ in many works), the ratio of the electron’s attributes as a mechanical gyroscope and a magnet. Because the electron’s charge is negative, $q = -e$, its magnetic dipole moment points opposite the spin vector. That means, for the electron, “spin up” has higher energy than “spin down” in the dipole-field potential energy $-\mathbf{m} \cdot \mathbf{B}$, of which more will be said shortly.

If the electric charge were distributed exactly the same as the mass then g_o would equal 1; therefore any deviation from unity suggests that the electric charge and mass are not identically distributed. The gyromagnetic ratios of the proton and neutron are about 5.586 and -3.826 respectively. Such numbers provide evidence that the proton and neutron are not elementary particles, but composites, in their case are built of quarks and gluons.

Uhlenbeck and Goudsmit initially suggested $g_o = 2$ for the electron, because that would fit the data available to them in 1926. For the

electron the Dirac equation predicts $g_0 = 2$, twice the value one would expect if the electron were a little ball that carried uniform charge density and uniform mass density with those densities distributed identically over the *same* volume. Evidently the distributions of mass and charge are *not* identical. It gets even more complicated: higher-order corrections in perturbative quantum mechanics (evaluating higher-order Feynman diagrams, through six and eight electron-photon vertices) show a power series in the fine structure constant,

$$g_0 = 2 + \alpha/\pi + \dots \quad (72)$$

where α denotes again the fine structure constant. High-order calculations from perturbation theory, along with precision measurements of $g_0 - 2$, provide sensitive tests of quantum electrodynamics (another story in its own right).[5] There's a lot of physics (not all of it understood) even in the humble electron! However, a discussion of those topics here would take us too far afield. I mention them only to show the connections between the path we follow here, and other routes that intersect ours in the rich conceptual space of physics. Here I want to talk about another kind of "g-factor" and two kinds of precession. Let's turn first to Larmor precession.

LARMOR PRECESSION

This is merely the usual precession that all gyroscopes experience when acted on by a torque, but we call it "Larmor precession" when the gyroscope happens to a magnetic dipole placed in a magnetic field. Let's review the precession of gyroscopes generically.

Let a top have spin angular momentum \mathbf{S} and let some torque $\mathbf{r} \times \mathbf{F}$ act on it thanks to a lever arm \mathbf{r} . Newton's Second Law in its rotational form, applied to the top, says

$$\mathbf{r} \times \mathbf{F} = d\mathbf{S}/dt. \quad (73)$$

Let the spin axis of the top be oriented at angle θ with respect to the z -axis, and let the force be parallel or antiparallel to the z -axis. For example, in the case of gravity, \mathbf{F} points vertically down and this will produce a torque on the top's center of mass given in magnitude by $rF \sin\theta$, and with a direction that points in the direction indicated by Fig. 2.

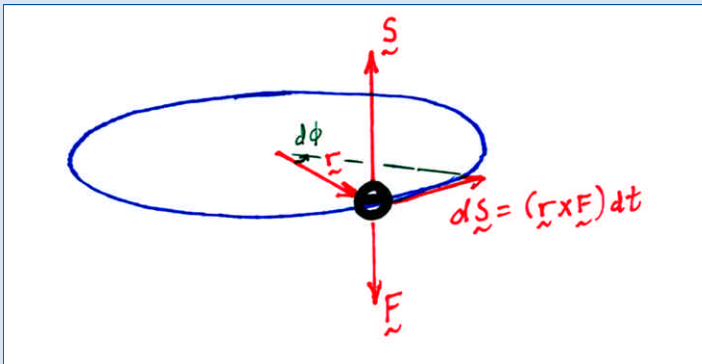


Fig. 2. Larmor Precession

The spin vector suffers a change $d\mathbf{S}$ in the same direction as the torque, and sweeps through the increment of magnitude $dS = S_z d\phi = S \sin\theta d\phi$. The $\sin\theta$ cancels out of Newton's Second Law, to give

$$rF = S d\phi/dt. \quad (74)$$

But $d\phi/dt$ is an angular frequency ω , and thus

$$\omega = rF/S. \quad (75)$$

In the case of a magnetic dipole \mathbf{m} that finds itself in some magnetic field \mathbf{B} , Newton's Second Law gives

$$\mathbf{m} \times \mathbf{B} = d\mathbf{S}/dt \quad (76)$$

where we recall that $\mathbf{m} = g_0 (q/2m)\mathbf{S}$. The same analysis of the torque and the precession response to it carries through, with $\mathbf{m}\mathbf{B}$ taking the place of rF . Now we have for the frequency of precession, in this context called the Larmor frequency,

$$\omega_{\text{Larmor}} = g_0 (|q|/2m) B. \quad (77)$$

(We might note that, once the precession gets underway, the center of mass orbits the z -axis, which means the top now carries some *orbital* angular momentum in addition to the spin. The rate of change of total angular momentum due to the torque gives "nutation" in addition to the precession, and another tangent that we don't have space here to follow.)

So, in the presence of a torque, a top in a gravitational field, or a magnetic dipole in a magnetic field, will precess. For a quantized angular momentum (whether it be \mathbf{L} or \mathbf{S} or \mathbf{J}), the vector precesses on the cone described earlier. Because the angle θ of the cone cancels out of the derivation of the precession frequency, it's the same for all θ . For a quantum spin, we can't track the tip of the spin vector; all we know is that it lies *somewhere* on the cone.

The interaction between a magnetic dipole and an applied magnetic field produces not only a torque but also a potential energy:

$$\begin{aligned} H' &= -\mathbf{m} \cdot \mathbf{B} \\ &= -\mathbf{m}\mathbf{B} \cos\theta. \end{aligned} \quad (78)$$

Recall that information about the electron's spin is carried by the two-entry row vector, or matrix χ , or "spinor" as it's called; in this language, any operations that manipulate the spin state must be represented with 2×2 matrices. For example, to measure the electron's interaction energy with the magnetic field, we have

$$\begin{aligned} H' &= -\mathbf{m} \cdot \mathbf{B} \\ &= -g_0 (q/2m) \mathbf{S} \cdot \mathbf{B} \\ &= -g_0 (q/2m) \frac{1}{2}\hbar \mathbf{s} \cdot \mathbf{B} \end{aligned} \quad (79)$$

Since we define the direction of \mathbf{B} to be the z -axis, as a matrix for manipulating the information in a spin state χ we have

$$H' = -\frac{1}{2}\hbar g_0 (q/2m) B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (80)$$

The eigenstates and eigenvalues of H' are of course χ_+ with eigenvalue $-\frac{1}{2}\hbar g_0 (q/2m)$ and χ_- with eigenvalue $+\frac{1}{2}\hbar g_0 (q/2m)$.

It should be noticed that, using the electron's mass and charge, if we set $g_0 = 2$ then $-\frac{1}{2}\hbar g_0 (q/2m) = +e\hbar/2m = 5.788 \times 10^{-5} eV/T$, the Bohr magneton, denoted μ_B in many books.

Suppose an electron (or any other spin- $1/2$ particle) finds itself in some generic spin state χ , which can be expanded in the χ_+ and χ_- basis:

$$\chi = u \chi_+ + v \chi_- \quad (81)$$

The coefficients u and v tell us, through $|u|^2$ and $|v|^2$, the probabilities that the electron will be found to have spin up or spin down respectively. These probabilities are *measured* by counting the fraction of particles, each in spin state χ , that, when the spin is measured, pops up as having spin up *or* spin down. This state χ evolves in time according to the Schrödinger equation,

$$H \chi = (-\hbar/i) \partial \chi / \partial t . \quad (82)$$

Split into its two components, and temporarily letting ϵ denote $-1/2\hbar g_o(q/2m)$ for brevity, this matrix equation becomes the pair of equations

$$\epsilon u = (-\hbar/i) \partial u / \partial t \quad (83a)$$

and

$$\epsilon v = (+\hbar/i) \partial v / \partial t \quad (83b)$$

with solutions

$$u(t) = u(0) e^{-i\epsilon t/\hbar} \quad (84a)$$

and

$$v(t) = v(0) e^{i\epsilon t/\hbar} \quad (84b)$$

Because the electron when hit with a magnetic field must line up into *either* the spin-up state *or* the spin-down state, at any time the probabilities sum to unity,

$$1 = |u|^2 + |v|^2 \quad (85)$$

so that we can always write

$$u(0) = \cos \delta \quad (86a)$$

and

$$v(0) = \sin \delta \quad (86b)$$

for some real number δ . How does this phase angle δ for the *spin state* relate to the angle θ in *physical space* that the spin vector makes with the magnetic field \mathbf{B} ? For the *classical* dipole, θ is determined from S_z according to

$$S_z = |\mathbf{S}| \cos \theta \quad (87)$$

where the *classical* $|\mathbf{S}|$ can be any non-negative value from the continuum that we want (it does not have to be some integer multiple of $1/2\hbar$). Let's calculate the expectation value of S_z for *quantized* spin. Here's its physical situation: we have an ensemble of electrons, each carefully placed somehow in the state χ , the same χ for each electron in the ensemble. We hit each electron with the magnetic field and measure whether it aligns itself "spin up" or "spin down," and collect statistics on the results. The numbers $|u|^2 = \cos^2 \delta$ and $|v|^2 = \sin^2 \delta$ will equal the fraction of electrons that have spin up and spin down respectively, if good agreement exists between theory and experiment (it does). A moment's reflection on visualizing the physical situation suggests that the average value of S_z for the ensemble will be 0 if half of the electrons have spin up and the other half have spin down, in which case $\delta = \pi/4$; yet for a classical macroscopic dipole with zero S_z we would have $\theta = \pi/2$. Let's see if such a correlation arises in the theory.

We must calculate the expectation value of S_z ,

$$\begin{aligned} \langle S_z \rangle &= \chi^\dagger S_z \chi \\ &= 1/2\hbar \chi^\dagger \sigma_z \chi \end{aligned} \quad (88)$$

where χ^\dagger denotes the row matrix obtained from the column matrix χ by transposing the latter and taking its complex conjugate. Carrying out the matrix multiplication, we find

$$\langle S_z \rangle = 1/2\hbar (\cos^2 \delta - \sin^2 \delta) \quad (89)$$

which might have been expected: the cosine squared gives the probability that the electron has spin up, and when it does have spin up then S_z equals $1/2\hbar$; and the sine squared gives the probability that the electron has spin down, in which case S_z equals $-1/2\hbar$. Thus

$$\begin{aligned} \langle S_z \rangle &= (\text{probability for spin up})(S_z \text{ with spin up}) \\ &\quad + (\text{probability for spin down})(S_z \text{ with spin down}). \end{aligned}$$

But notice the crucial minus sign between the cosine squared and sine squared terms. It reminds us of the existence of a trig identity,

$$\cos^2 \delta - \sin^2 \delta = \cos(2\delta) \quad (90)$$

and thus

$$\langle S_z \rangle = 1/2\hbar \cos(2\delta) . \quad (91)$$

When we compare the quantum spin *state* to the *direction* of the classical spin, we see that the phase δ and the physical angle θ are related by

$$\delta = 1/2\theta . \quad (92)$$

This *phase* δ of the spin state is *half* the angle θ in physical space made by the spin vector relative to the z -quantization axis! Such a result formalizes our anticipation: if half the electrons have spin up and half have spin down, then their respective probabilities are $1/2$ each, which requires $\delta = 45^\circ$; this means in turn that the ensemble of electrons behaves *statistically* like a single classical magnetic dipole tipped over at $\theta = 90^\circ$, with zero z -component for the spin.

Curiously, this means that if we envision a spin vector that we can rotate through any angle we want, when we start out at $\theta = 0$ and rotate through 180° we convert a (classical) $S_z = +S$ into $S_z = -S$. Now keep on rotating the vector on around, past 180° on back to 360° . The classical spin vector has been restored to its original state ($\Delta\theta = 2\pi$), but the quantum spin phase δ has gone through only half a cycle ($\Delta\delta = \pi$)! In other words, to restore the *quantum state* back to its original value we have to rotate the spin vector through 4π radians of *physical space*, two full revolutions![6]

OK then, we have found the expectation value of S_z ; in particular, in terms of the spatial angle θ ,

$$\langle S_z \rangle = 1/2\hbar \cos \theta . \quad (93)$$

S_z is independent of time, so long as the magnetic field is static. But we recall that the x and y components of \mathbf{S} precess around the cone; they are time-dependent, even in a static field. Let's see if this intuition stands up to explicit calculation. Straightforward matrix algebra gives

$$\begin{aligned}
\langle S_x \rangle &= \chi^\dagger S_x \chi \\
&= \frac{1}{2}\hbar \chi^\dagger \sigma_x \chi \\
&= \frac{1}{2}\hbar \sin\theta \cos\omega t
\end{aligned} \tag{94}$$

and likewise

$$\langle S_y \rangle = -\frac{1}{2}\hbar \sin\theta \sin\omega t \tag{95}$$

where ω denotes the Larmor frequency. We can see that the three components of \mathbf{S} , written in terms of θ and ωt , describe a vector whose z -component stays fixed but whose x and y coordinates move in a circle. Thus \mathbf{S} precesses on the cone at the Larmor frequency.

It takes a magnetic field to reveal the spin of an electron. The magnetic field's direction *defines* the z -axis for quantization. The magnetic field may come from an external source that *we* supply. Let's call such a field the "external magnetic field," \mathbf{B}_{ext} . In addition, whenever an electron finds itself in orbit about the nucleus of an atom, it will automatically be inside a magnetic field produced by the atom itself, thanks to the relative motion between electron and nucleus. Let's call such a field the "internal field," \mathbf{B}_{int} . We cannot control \mathbf{B}_{int} ; it's *always* there inside the atom. So the interaction between the electron and a magnetic field includes two possibilities:

$$\begin{aligned}
H' &= H'_{\text{ext}} + H'_{\text{int}} \\
&= -\mathbf{m}_1 \cdot \mathbf{B}_{\text{ext}} - \mathbf{m}_2 \cdot \mathbf{B}_{\text{int}}
\end{aligned} \tag{96}$$

where \mathbf{m}_1 and \mathbf{m}_2 are, respectively, whatever magnetic dipole moments are relevant in the cases of the externally-supplied and the internally-supplied fields. These options lead us, in the first case, to the Zeeman effect, with the Landé g -factor; and in the second case, to the spin-orbit interaction with Thomas Precession. We will consider them in turn.

THE ZEEMAN EFFECT

In the Zeeman effect we consider the influence of an externally-supplied magnetic field on an electron that's bound to an atom, say a hydrogen atom. We set up the Zeeman effect by placing the atom between the poles of a strong magnet. Thus the external field \mathbf{B}_{ext} is some steady, uniform field \mathbf{B}_o whose direction defines our z -axis. This field is under the control of the experimenter.

For \mathbf{m}_1 we have of course the contribution from spin, Eq. (71), that we denote in this section as

$$\mathbf{m}_S = g_o (q/2m)\mathbf{S} . \tag{97}$$

But as the electron orbits the nucleus, its orbital angular momentum also describes an electric current with its own orbital magnetic dipole moment,

$$\mathbf{m}_L = g'_o (q/2m)\mathbf{L} . \tag{98}$$

For the purposes of computing its *orbital* angular momentum, the electron is indistinguishable from a point mass, leading us to expect that $g'_o = 1$. A general physics calculation of μ_L supports this notion. As the electron orbits, in particle language consider a circular path of radius r with the electron moving along it at the speed v . The dipole moment of a current loop equals the current times the area,

$$\mu_L = (q/T)\pi r^2 \tag{99}$$

where T is the period of one orbit. But the electron's speed is given by $v = 2\pi r/T$, so that

$$\mu_L = (q/2m) L . \tag{100}$$

If $q < 0$ then \mathbf{m}_L points opposite \mathbf{L} ; if $q > 0$ then \mathbf{m}_L and \mathbf{L} point in the same direction; in either case $\mathbf{m}_L = (q/2m)\mathbf{L}$, and the g -factor for the gyromagnetic ratio for *orbital* motion is unity. Returning to the externally-sourced interaction between the electron and the magnetic field, and denoting H'_{ext} as H'_{Zeeman} , we have

$$H'_{\text{Zeeman}} = -(q/2m) (\mathbf{L} + g_o\mathbf{S}) \cdot \mathbf{B}_o . \tag{101}$$

Our task is to calculate the expectation value of H'_{Zeeman} whenever the electron finds itself in, say, some hydrogen atom state $|\psi\rangle$ (which may not be one of the hydrogen atom eigenstates, but can always be written as a superposition of them). This means we must evaluate

$$\langle H'_{\text{ext}} \rangle = \langle \psi | H'_{\text{Zeeman}} | \psi \rangle . \tag{102}$$

We will see that the result, at the end of the day, can be written in the form that resembles the original interaction,

$$\langle H'_{\text{Zeeman}} \rangle = -g_{\text{Landé}} (q/2m) \langle \mathbf{J} \rangle \cdot \mathbf{B}_o \tag{103}$$

where the Landé g -factor, $g_{\text{Landé}}$ is built from g_o and the angular momentum quantum numbers that belong to $|\psi\rangle$, as we now demonstrate.

Our task requires us to calculate

$$\langle H'_{\text{Zeeman}} \rangle = -(q/2m) \langle \mathbf{L} + g_o\mathbf{S} \rangle \cdot \mathbf{B}_o . \tag{104}$$

We recall that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and $g_o = 2 + \epsilon$ where $\epsilon = \alpha/\pi + \text{higher-order terms in } \alpha$. Splitting g_o this way allows us to write

$$\langle H'_{\text{Zeeman}} \rangle = -(q/2m) \langle \mathbf{J} + (1 + \epsilon)\mathbf{S} \rangle \cdot \mathbf{B}_o . \tag{105}$$

When we form the new vector \mathbf{J} from \mathbf{L} and \mathbf{S} , we realize from our generic discussion earlier that the information carried by a quantum state of angular momentum \mathbf{J} can be encoded in two sets of numbers: j and m_j , that label the state $|j, m_j\rangle$, such that

$$J^2 |j, m_j\rangle = \hbar^2 j(j+1) |j, m_j\rangle \tag{106}$$

and

$$J_z |j, m_j\rangle = \hbar m_j |j, m_j\rangle . \tag{107}$$

The values of j take on these possibilities:

$$j = \ell + s, \ell + s - 1, \dots, |\ell - s| . \tag{108}$$

and for each j ,

$$m_j = j, j - 1, \dots, -j . \tag{109}$$

When $s = 1/2$ then for a given ℓ , j may assume the two values $\ell \pm 1/2$.

Under these circumstances the following commutation relations hold:

$$[J^2, J_z] = 0, \quad (110)$$

$$[J^2, L^2] = 0, \quad (111)$$

$$[J^2, S^2] = 0, \quad (112)$$

but the commutators of L_z and S_z with components of \mathbf{J} or J^2 are *not* zero. The upshot of all this commutator algebra means that the quantities *simultaneously observable* are the magnitudes of \mathbf{J} , \mathbf{L} , and \mathbf{S} (through their squares) and the z -component of \mathbf{J} . But the z -components of \mathbf{L} and \mathbf{S} (and the x and y components of \mathbf{J} , \mathbf{L} , and \mathbf{S} , as before) are *not* observable. However, our expectation value requires us to deal with $\mathbf{J} + (1+\epsilon)\mathbf{S}$. This term gets dotted into \mathbf{B}_o , and thus we have to evaluate $J_z + (1+\epsilon)S_z$. J_z is no problem; we have a good quantum number for it. But the state of coupled angular momentum has no unique quantum number for S_z . See Fig. 3

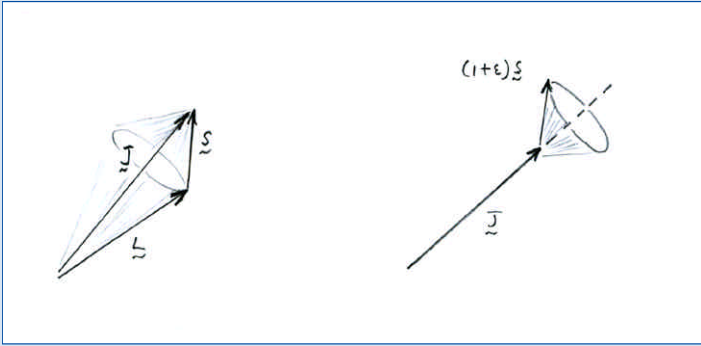


Fig. 3. $\mathbf{J} = \mathbf{L} + \mathbf{S}$ (left), and $\mathbf{J} + (1+\epsilon)\mathbf{S}$ (right).

It will however be noticed that, after several Larmor periods, the time-average of $(1+\epsilon)\mathbf{S}$ will point in the same direction as \mathbf{J} . Let $\langle \mathbf{S} \rangle$ denote the time-average of a periodic motion. Figure 3 suggests

$$\langle \mathbf{S} \rangle = (\mathbf{S} \cdot \mathbf{J}) \mathbf{J} / J^2. \quad (113)$$

So although we can't measure S_z directly, we can measure the time average of the expectation value of H'_{Zeeman} :

$$\langle H'_{\text{Zeeman}} \rangle = -(q/2m) [1 + (1+\epsilon)(\mathbf{S} \cdot \mathbf{J})/J^2] (\mathbf{J} \cdot \mathbf{B}_o). \quad (114)$$

For a hydrogen-like atom, the $|\psi\rangle$ that forms the state of definite \mathbf{J} , \mathbf{L} , and \mathbf{S} , denoted $|n j m_j \ell s\rangle$, will be superposition of the states $|\ell m_\ell s m_s\rangle$ [7]. Because $\mathbf{L} = \mathbf{J} - \mathbf{S}$, we can find $\mathbf{J} \cdot \mathbf{S}$ in terms of the knowable quantum numbers of $|n j m_j \ell s\rangle$:

$$\mathbf{J} \cdot \mathbf{S} = \frac{1}{2}(J^2 - L^2 + S^2). \quad (115)$$

Therefore

$$\langle H'_{\text{Zeeman}} \rangle = g_{\text{Landé}} (q/2m) \langle \mathbf{J} \cdot \mathbf{B}_o \rangle \quad (116)$$

where we have introduced the Landé g -factor,

$$g_{\text{Landé}} = 1 + \frac{(1+\epsilon)[j(j+1) - \ell(\ell+1) + 3/4]}{2j(j+1)}. \quad (117)$$

The Landé g -factor can be written in terms of the gyromagnetic g -factor, showing the connection between them:

$$g_{\text{Landé}} = 1 + \frac{(g_o - 1)[j(j+1) - \ell(\ell+1) + 3/4]}{2j(j+1)}. \quad (118)$$

The gyromagnetic factor g_o is a property of the *electron*; the Landé g -factor is a property of the electron *and* the *state of the atom* in which it happens to reside.

THE SPIN-ORBIT INTERACTION

When an electron orbits a nucleus to form an atom, the relative motion between the nucleus and the electron generates a magnetic field. We must evaluate

$$H'_{\text{int}} = -\mathbf{m}_2 \cdot \mathbf{B}_{\text{int}} \quad (119)$$

The dipole moment \mathbf{m}_2 is that due to the electron's spin, our familiar friend $\mathbf{m} = g_o (q/2m)\mathbf{S}$. There is no \mathbf{m}_L contribution here because, as a scalar, H'_{int} has the same numerical value in all reference frames, and it's convenient to solve the problem in the electron's frame (which sees the proton orbit the electron). When you are at rest relative to the electron, it has zero orbital angular momentum!

For the internal magnetic field, we appeal to the Special Theory of Relativity. In 1905 Albert Einstein first worked out the relativity of electric and magnetic fields between two *inertial* frames.[8] Consider a "Lab Frame" (with unprimed quantities) and a "Rocket Frame" (with primed variables) that glides with uniform velocity \mathbf{v} relative to the lab.[9] The frame in which the proton is at rest will be our Lab Frame, and the frame where the electron is at rest will be our Rocket Frame (or, we would like it to be!). In the proton frame, let the electric field be \mathbf{E} and let the magnetic field be \mathbf{B} . An observer in the Rocket Frame sees the electric and magnetic fields \mathbf{E}' and \mathbf{B}' , where according to Einstein,

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{\gamma^2 \mathbf{v}(\mathbf{v} \cdot \mathbf{E})}{(\gamma+1)c^2} \quad (120)$$

and

$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2) - \frac{\gamma^2 \mathbf{v}(\mathbf{v} \cdot \mathbf{B})}{(\gamma+1)c^2}. \quad (121)$$

In these formulas, γ denotes the kinematic factor familiar from Special Relativity:

$$\gamma = (1 - v^2/c^2)^{-1/2}. \quad (122)$$

In the problem before us, the proton produces an electric field \mathbf{E} but no magnetic field in its own frame. Therefore $\mathbf{B} = 0$ and

$$\mathbf{B}' = -\gamma \mathbf{v} \times \mathbf{E}/c^2 \quad (123)$$

describes (almost!) the magnetic field "seen" by the electron. Are we ready to import this result into Eq. (119) to assume its role for \mathbf{B}_{int} ? Not quite; the transformation we have used for the electric and magnetic fields hold between *inertial* reference frames, where one frame moves with *constant* velocity relative to the other. However, our electron *orbits* the proton; hence the electron finds itself in an accelerated frame of reference. Happily, the correction we need for \mathbf{B}' is an *additive* correction, as Thomas showed us in 1927. To Thomas' correction we now turn.

Consider a rotating reference frame—a Carousel Frame—that rotates with angular velocity ω_T relative to the Lab Frame (this ω_T is *not* the Larmor frequency). Let a particle's position \mathbf{r} change with rate $(d\mathbf{r}/dt)_{\text{rotating}}$ relative to the rotating frame; for instance, if \mathbf{r} locates the position of a ladybug in the Carousel Frame, then $(d\mathbf{r}/dt)_{\text{rotating}}$ denotes the velocity of the ladybug *relative to the carousel* as she walks across the carousel floor. Even if she does *not* walk on the carousel, the ladybug still gets carried about, relative to the Lab Frame, with velocity $\omega_T \times \mathbf{r}$ because of the carousel's rotation. So if the ladybug gets carried by the carousel *and* walks across it, then her velocity relative to the Lab is [10]

$$(d\mathbf{r}/dt)_{\text{lab}} = (d\mathbf{r}/dt)_{\text{rotating}} + \omega_T \times \mathbf{r} . \quad (124)$$

This result generalizes: for any vector \mathbf{f} (all vectors *transform* the same as position vectors),

$$(d\mathbf{f}/dt)_{\text{lab}} = (d\mathbf{f}/dt)_{\text{rotating}} - \mathbf{f} \times \omega_T . \quad (125)$$

(It will be noticed that Newtonian relativity is being used here in adding the velocities, and in using the same time interval dt for both frames. However, we shall see that we may set the relativistic kinematic factor $\gamma \approx 1$. We especially need $\gamma+1$ for the evaluation of the Thomas frequency ω_T , as we shall see).

Now let's apply Eq. (125) to the relativity of the rate of change of the spin vector between a non-rotating and a rotating frame:

$$d\mathbf{S}/dt(\text{lab frame}) = d\mathbf{S}/dt(\text{rotating frame}) - \mathbf{S} \times \omega_T . \quad (126)$$

Because the rate of angular momentum change is produced by the magnetic torque, this says

$$\mathbf{m} \times \mathbf{B}_{\text{lab}} = \mathbf{m} \times \mathbf{B}' - \mathbf{S} \times \omega_T \quad (127)$$

But from Eq. (71) we have $\mathbf{S} = (2m/q)(1/g_o)\boldsymbol{\mu}$, and therefore

$$\mathbf{m} \times \mathbf{B}_{\text{lab}} = \mathbf{m} \times [\mathbf{B}' - (2m/q)(1/g_o)\omega_T] . \quad (128)$$

The dipole-field interaction energy will therefore be

$$-\mathbf{m} \cdot \mathbf{B} = -\mathbf{m} \cdot [\mathbf{B}' - (2m/q)(1/g_o)\omega_T] . \quad (129)$$

This suggests we identify our rotating-frame-corrected \mathbf{B}' as

$$\mathbf{B}'_{\text{corrected}} = \mathbf{B}' - (2m/q)(1/g_o)\omega_T . \quad (130)$$

Using a combination of the Lorentz transformation (to give the electron a velocity relative to the proton) and rotations (to have that velocity orbit the proton), Thomas was able to show in 1927 that [11]

$$\omega_T = -\frac{\gamma^2 (\mathbf{v} \times \mathbf{a})}{(\gamma+1)c^2} \quad (131)$$

where \mathbf{a} is the electron's acceleration as measured in the lab (proton) frame. For this acceleration we may write (again, falling back on Newtonian theory in this part of the problem)

$$\mathbf{a} = q\mathbf{E}/m \quad (132)$$

so that

$$\omega_T = -\frac{\gamma^2 (q/m) (\mathbf{v} \times \mathbf{E})}{(\gamma+1)c^2} \quad (133)$$

Therefore, with Eqs. (123) and (133), $\mathbf{B}'_{\text{corrected}}$ becomes

$$\begin{aligned} \mathbf{B}'_{\text{corrected}} &= -\gamma \mathbf{v} \times \mathbf{E}/c^2 + (2/g_o) \frac{\gamma^2 (\mathbf{v} \times \mathbf{E})}{(\gamma+1)c^2} \\ &= -(\gamma/c^2) [1 - \frac{(2/g_o)\gamma}{\gamma+1}] (\mathbf{v} \times \mathbf{E}) \end{aligned} \quad (134)$$

When the electric field has spherical symmetry and thus a central potential $\Phi = \Phi(r)$, so that

$$\mathbf{E} = - (d\Phi/dr) \mathbf{r}/r , \quad (135)$$

then

$$\begin{aligned} \mathbf{v} \times \mathbf{E} &= - (\mathbf{v} \times \mathbf{r})(d\Phi/dr)(1/r) \\ &= \mathbf{L} (d\Phi/dr)(1/mr) . \end{aligned} \quad (136)$$

Now we can return to Eq. (119) and resume the calculation of H'_{int} :

$$H'_{\text{int}} = g_o(q/2m)(\gamma/rmc^2) [1 - \frac{(2/g_o)\gamma}{\gamma+1}] (d\Phi/dr) \mathbf{S} \cdot \mathbf{L} \quad (137)$$

The $\mathbf{S} \cdot \mathbf{L}$ term endows this interaction with the name “spin-orbit interaction,” henceforth denoted H'_{LS} . If the potential is that of the proton's Coulomb field, then $\Phi = ke/r$ and $d\Phi/dr = -ke/r^2$. Using $q = -e$ for the electron's charge, we have

$$H'_{\text{LS}} = g_o(ke^2/2m^2 c^2)(\gamma/r^3)[1 - \frac{(2/g_o)\gamma}{\gamma+1}] \mathbf{S} \cdot \mathbf{L} \quad (138)$$

Let's estimate the value of γ , at least to back-of-the-envelope accuracy. In the simple Bohr model of the hydrogen atom, the electron has energy

$$E = \frac{1}{2}mv^2 - ke^2/r \quad (139)$$

while $F = ma$ gives

$$ke^2/r^2 = mv^2/r . \quad (140)$$

Combining Eqs. (139) and (140) shows

$$E = -\frac{1}{2}mv^2 . \quad (141)$$

In the ground state, we have a numerical value for E of -13.6 eV. Writing the electron mass as $m \approx \frac{1}{2} \text{MeV}/c^2$ tells us that $v^2 \approx 5 \times 10^{-5} c^2$, so that with the help of a binomial expansion we find

$$\gamma \approx 1 - \frac{1}{2} v^2/c^2 = 1 - 2.5 \times 10^{-5} . \quad (142)$$

Therefore we make an error of less than a part in a hundred thousand if we approximate $\gamma \approx 1$. If we further approximate g_o with 2, then

$$H'_{\text{LS}} \approx \frac{1}{2}(ke^2/m^2 c^2) (1/r^3) \mathbf{S} \cdot \mathbf{L} \quad (143)$$

Whether or not one makes these approximations, we calculate $\mathbf{S} \cdot \mathbf{L}$ from squaring $\mathbf{J} = \mathbf{L} + \mathbf{S}$:

$$\mathbf{S} \cdot \mathbf{L} = \frac{1}{2}(J^2 - L^2 - S^2) . \quad (144)$$

When sandwiched between states of angular momentum \mathbf{J} that have been formed from a superposition of states of angular momenta \mathbf{L} and \mathbf{S} , we find

$$\begin{aligned} \langle H'_{LS} \rangle &= \frac{1}{2}(ke^2/m^2c^2) \langle (1/r^3) \mathbf{S} \cdot \mathbf{L} \rangle \\ &= \frac{1}{4}(ke^2 \hbar^2/m^2 c^2) [j(j+1) - \ell(\ell+1) - \frac{3}{4}] \\ &\times \langle n j m_j \ell s | 1/r^3 | n j m_j \ell s \rangle . \end{aligned} \quad (145)$$

There are two possible values for this result, because $j = \ell \pm \frac{1}{2}$. But we'll have to leave the problem here for now. To finish the problem one must evaluate the expectation value of $1/r^3$ when it's sandwiched between hydrogen atom states. Techniques exist for doing so.[12]

FERMIONS, BOSONS, AND SYSTEMS OF IDENTICAL PARTICLES

Any two electrons are *identical*; in a shell game with them they *cannot* be distinguished, *in principle*. If two electrons are switched there's *no way to tell any difference* between the original and exchanged configuration. That's the *definition* of identical particles: "No difference!" between the "original" and "exchanged" configurations. However, because two ways exist to place a pair of electrons—original and switched—then according to the rules of quantum mechanics,[13] *the* quantum state of the two-particle system must be formed by summing over all the possible combinations. Even though we can't tell them apart, quantum mechanics *requires* us to sum over "original" and "exchanged" configurations. This has profoundly observable consequences.

Let's consider a quantum system that has two identical particles (a pair of electrons, neutrinos, neutral pions, or whatever), particle 1 and particle 2, with one of them in some one-particle state a and the other in some one-particle state b . The *two*-particle quantum system ψ can be built out of these one-particle states, but we must include both the the "original" and "exchanged" configurations:

$$\psi(1,2) = c_{\text{original}} a(1)b(2) + c_{\text{exchange}} a(2)b(1) . \quad (146)$$

The probabilities of finding the system in the original or in the exchanged configurations sum to unity,

$$|c_{\text{original}}|^2 + |c_{\text{exchange}}|^2 = 1 . \quad (147)$$

Because the particles are identical, these probabilities must be equal:

$$|c_{\text{original}}|^2 = |c_{\text{exchange}}|^2 \quad (148)$$

Therefore, the two coefficients differ at most by a phase:

$$\begin{aligned} c_{\text{exchange}} &= c_{\text{original}} e^{i\delta} \\ &= 2^{-\frac{1}{2}} e^{i\delta} . \end{aligned} \quad (149)$$

So the two-particle state becomes, for *identical* particles,

$$\psi(1,2) = 2^{-\frac{1}{2}} [a(1)b(2) + e^{i\delta} a(2)b(1)] . \quad (150)$$

Let's introduce the "exchange operator," \mathcal{E} , defined by

$$\mathcal{E}\psi(1,2) = \psi(2,1) . \quad (151)$$

This operator switches particles 1 for 2 and *vice versa*. Because the two particles are identical, the system's physics will be the same before and after the exchange operator sweeps through the system and interchanges the particles. This means that the exchange operator commutes with the two-particle system's Hamiltonian:

$$[\mathcal{E}, H] = 0 . \quad (152)$$

Therefore $\psi(1,2)$ is *simultaneously* an eigenstate of the Hamiltonian with some energy eigenvalue E , and also the eigenstate of the exchange operator with some eigenvalue η :

$$H\psi(1,2) = E\psi(1,2) \quad (153)$$

and

$$\mathcal{E}\psi(1,2) = \eta\psi(1,2) . \quad (154)$$

Let's find η : If we operate on Eq. (154) with \mathcal{E} a second time, we have $\mathcal{E}^2 \psi(1,2) = \eta^2 \psi(1,2)$. But the left-hand side exchanges particles 1 and 2 *twice*:

$$\mathcal{E}^2 \psi(1,2) = \mathcal{E}\psi(2,1) = \psi(1,2) \quad (155)$$

and therefore $\eta^2 = 1$ so that

$$\eta = \pm 1 . \quad (156)$$

Return to Eq. (150), and operate on it with the exchange operator, then develop the right-hand side:

$$\begin{aligned} \mathcal{E}\psi(1,2) &= 2^{-\frac{1}{2}} \mathcal{E}[a(1)b(2) + e^{i\delta} a(2)b(1)] \\ &= 2^{-\frac{1}{2}} [a(2)b(1) + e^{i\delta} a(1)b(2)] \\ &= 2^{-\frac{1}{2}} e^{i\delta} [a(1)b(2) + e^{-i\delta} a(2)b(1)] \end{aligned} \quad (157)$$

Recalling that $\mathcal{E}\psi(1,2) = \pm\psi(1,2)$, the left-hand side of Eq. (157) may be written

$$\mathcal{E}\psi(1,2) = \pm 2^{-\frac{1}{2}} [a(1)b(2) + e^{i\delta} a(2)b(1)] . \quad (158)$$

Comparing the right-hand sides of Eqs. (157) and (158), self-consistency requires that

$$e^{i\delta} = \pm 1 . \quad (159)$$

Placing this back in Eq. (150), we have a result that carries profound consequences:

$$\psi(1,2) = 2^{-\frac{1}{2}} [a(1)b(2) \pm a(2)b(1)] . \quad (160)$$

Here we confront a curious result that quantum mechanics has placed in our path: for a system of two identical particles, the exchanged amplitude combines additively with the original amplitude—but with *either* a plus sign *or* a minus sign. That's a *logical consequence* of the simple rules of quantum mechanics and of the particles being identical.

But this result leads to a deeper question: *What determines the choice of the plus sign or the minus sign?* One might imagine all kinds of possibilities: for example, one might imagine a universe where identical neutral particles combine with the plus sign but identical charged particles combine with the minus sign; or identical particles with masses above some critical value use one sign but those

with masses below that threshold use the other sign. But these are not the answer. The answer that Nature gives us is expressed in the celebrated Spin-Statistics Theorem: Identical *fermions* (that have $s = 1/2$ or $3/2$ or $5/2, \dots$) add with the *minus* sign, and identical *bosons* (for which $s = 0$ or 1 or $2, \dots$) add with the *plus* sign. The consequences are *enormous*.

First, consider fermions, which use the minus sign. We see immediately that if states a and b are the *same* state, so that $a = b$, then $\psi(1,2) = 0$. Therefore, no two identical fermions *can ever* be in the same single-particle state! This corollary to the Spin-Statistics Theorem is called the “Pauli Exclusion Principle.” Its consequence follows: In the “ground state” of a multi-electron atom, the electrons cannot all congregate in the $1s$ orbital, as we would expect them to do from energy minimization considerations alone. Instead, because of the Spin-Statistics Theorem, each orbital can hold a maximum of two electrons, and only then if their spins are opposite, so that no two electrons have the same quantum numbers for their single-particle states $|n \ell m_\ell s m_s\rangle$. Therefore, the electrons in an atom are obliged to “stack up,” populating the higher-energy orbitals. This feature of the world leads to the existence of atoms and molecules and matter and chemistry and biology and life! In even the most extreme situations, this Pauli Exclusion blocking by electrons prevents further gravitational collapse of a white dwarf star if the dwarf’s mass lies below the Chandrasakhar limit of 1.4 solar masses; should that limit be exceeded, the electrons are driven into the protons to form a neutron star (with a burst of neutrinos). The neutrons, being fermions, are themselves subject to Pauli blocking which halts further collapse unless the mass is *so* great—the compact object having more than three times the mass of the sun—that it undergoes ultimate gravitational collapse to a black hole.

Now consider identical bosons. Not only *may* more than one of them be in the same state; they “like” to coalesce into the same state. This, too, has consequences. For instance, photons have spin 1, but we would see dimly in a world of widely separated photons. But being bosons, the individual photons “want to” coalesce into the same state having the same energy and polarization, and *voilà!*—we have macroscopic light waves! Helium-4, too, when cooled to near absolute zero, exhibits strange macroscopic behavior because the atoms, as bosons, coalesce into the same quantum state.

Without this amazing feature of Nature that we express in the language of the Spin-Statistics Theorem, the universe would be radically different.

The little demonstration offered above shows that pairs of identical particles *must* combine with either the plus or the minus sign. The Spin-Statistics Theorem, which tells us why this sign gets chosen according to the particle’s spin, was proved by Wolfgang Pauli in 1940,[14] using complicated arguments from relativistic quantum field theory. Pauli gave us a *logical* if esoteric proof of the connection, but it cannot offer the kind of insight where we can help students see it for themselves and come to say, “Of course, it *has* to be that way for the following simple reason...” It’s not a stretch to suppose that the Spin-Statistics Theorem depends on the behavior of particles under a rotation of the reference frame, because at least in the case of spin- $1/2$ particles, flipping a pair of opposite spins symmetrically placed on the x -axis gives the same result as rotating the reference frame 180° about the z -axis. But there’s more to it than that.

As Richard Feynman famously observed, the absence of a proof that uses only elementary concepts probably means we do not *really understand* the Theorem well. In the *Feynman Lectures on Physics*, he asked:

Why is it that particles with half-integral spin are Fermi particles whose amplitudes add with a minus sign, whereas particles with integral spin are Bose particles whose amplitudes add with the positive sign? We apologize for the fact that we cannot give you an elementary explanation. An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity. He has shown that the two must necessarily go together, but we have not been able to find a way of reproducing his arguments on an elementary level. It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation. The explanation is deep down in relativistic quantum mechanics. This probably means that we do not have a complete understanding of the fundamental principle involved.[15]

One wonders,[16] has anyone made progress towards an “elementary” demonstration of the Spin-Statistics Theorem? A lively discussion on this point has ensued in recent years.[17] The state of our present understanding is wonderfully reviewed, further insights into the question are probed, and remedies suggested, in a book dedicated to this purpose by E. C. G. Sudarshan and Ian Duck. They open their book with these lines:

Everyone knows the Spin-Statistics Theorem but no one understands it. The key word is “understand.” The Spin-Statistics Theorem...stands as a fact of nature. The question is whether physics contains this fact, and if so how this comes about; or whether physics is merely consistent with the Spin-Statistics Theorem and that some deeper explanation exists. The Spin-Statistics Theorem could conceivably be an essential ingredient of a more fundamental view of the world...

With such a point of view forced upon us, we should modify the meaning of “understand”, and at the same time reduce our expectations of any proof of the Spin-Statistics Theorem. What is proved...is that the existing theory is consistent with the spin-statistics relation. What is not demonstrated is a reason for the spin-statistics relation...

Must we reduce our demands on physics to require only consistency? Does an understanding of the “Why?” of the spin-statistics relation have no direct answer in physics? Or must physics be formulated to include it? The Pauli result does not explain the spin-statistics relation and cannot...because the consistency of relativistic quantum mechanics and quantum field theory with the Pauli Exclusion Principle has every reason to be as complicated as these subjects are, not as simple and direct as the Pauli Exclusion Principle itself...[18]

When we entered college or university, we may have thought we would walk out four or five years later with Answers to Life’s Questions. We quickly discover, however, that we do well to understand the Questions. Sometimes, understanding the question is the best we can do! That, in itself, represents no small achievement.

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